Notes.

(a) You may freely use any result proved in class or in the textbook unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) $\mathbb{R}$ = real numbers.

(c) All the manifolds here are assumed to be submanifolds in some Euclidean space $\mathbb{R}^N$.

Part A : Do any 2 of the following 3 questions.

A1. [20 points] Let $f\colon X \to Y$ be a smooth bijection of boundaryless manifolds. Prove that there exists a non-empty open subset $X_1 \subset X$ such that $f\big|_{X_1}$ is a diffeomorphism. Further prove that X_1 may be chosen such that $f(X_1)$ is dense in Y.

A2. [20 points] Define what it means for a smooth function $f\colon X \to \mathbb{R}$ to be a Morse function on the boundaryless manifold X. Explain why the critical points of f form an isolated subset of X.

A3. [20 points] Prove that any boundaryless k-manifold X admits an immersion to $\mathbb{R}^{2k}$. (Hint: Project from higher dimensions by avoiding the tangential directions.)

Part B : Do any 4 of the following 5 questions.

B1. [15 points] Let $S \subset \mathbb{R}^n$ be a boundaryless manifold. Suppose S intersects ∂H^n transversally where $H^n \subset \mathbb{R}^n$ is the half plane where the last coordinate is non-negative. Prove that $S \cap H^n$ is a manifold with boundary $S \cap \partial H^n$.

B2. [15 points] Let X be a compact manifold with boundary ∂X. Let $g\colon \partial X \to \partial X$ be a diffeomorphism. Prove that g does not extend to a smooth map $G\colon X \to \partial X$.

B3. [15 points] Let $X, Y \subset \mathbb{R}^N$ be boundaryless submanifolds. Prove that for almost every $a \in \mathbb{R}^N$, the translate $X + a$ intersects Y transversally.

B4. [15 points] Define the following terms and along with it specify the additional assumptions that the manifolds and maps may need to satisfy for the purpose of the definition:

(i) The mod 2 intersection number $I_2(X, Z)$ for submanifolds $X, Z \subset Y$.

(ii) The mod 2 degree $\deg_2(f)$ of a smooth map $f\colon X \to Y$.

(iii) The winding number $W_2(f, z)$ of a smooth map $f\colon X \to Y$ around a point $z \in Y$.

B5. [15 points] Let X be a compact boundaryless k-manifold in $\mathbb{R}^{2k}$.

(i) Prove that $I_2(X, X)$ (the mod 2 self-intersection number of X in $\mathbb{R}^{2k}$) is 0.

(ii) Prove that there is a boundaryless manifold Y and diffeomorphisms $X \xrightarrow{\sim} X_i \subset Y$ for $i = 1, 2$ such that $I_2(X_1, X_2) = 1$.